

Reflexive spaces. (Important theorems)

Definition: — The mapping of L into L^{**} defined by the linear transformation J is called the canonical mapping of L into L^{**} .

Def: — If range of J i.e. $R(J) = L^{**}$, then the linear space L is said to be algebraically reflexive.

Theorem: — Every finite dimensional linear space is algebraically reflexive.

Proof: — Let L be a finite dimensional linear space.

We know that the mapping J from L into L^{**} is a linear transformation from L into L^{**} and J is one-one. Since L is finite dimensional, therefore $\dim L = \dim L^* = \dim L^{**}$. Hence J is one-one implies J must be onto. Therefore $R(J) = L^{**}$ and thus L is algebraically reflexive.

Theorem: — If L is an infinite-dimensional linear space, then it is not algebraically reflexive. Consequently a linear space is algebraically reflexive if and only if it is finite dimensional.

Proof: — Let $B = \{x_i; i \in I\}$ be a Hamel basis for L .

Since L is finite dimensional, therefore the index set I is an infinite set and $x_i \neq x_j$ if $i \neq j$. Define $f_i \in L^*$ by $f_i(x_i) = 1, f_i(x_j) = 0$ if $i \neq j$. We claim that the set $\{f_i; i \in I\}$ is a linearly independent subset of L^* .

For suppose that $\{f_{i_1}, f_{i_2}, \dots, f_{i_n}\}$ is any finite subset of the set $\{f_i; i \in I\}$.

Let $\alpha_1, \dots, \alpha_n$ be scalars such that

$$\alpha_1 f_{i_1} + \dots + \alpha_n f_{i_n} = \hat{0} \text{ (origin of } L^*)$$

$$\Rightarrow (\alpha_1 f_{i_1} + \dots + \alpha_n f_{i_n})(x) = \hat{0}(x) \forall x \in L$$

$$\Rightarrow \alpha_1 f_{i_1}(x) + \dots + \alpha_n f_{i_n}(x) = 0 \forall x \in L$$

$$\Rightarrow \alpha_\mu f_{i_\mu}(x_{i_\mu}) + \dots + \alpha_n f_{i_n}(x_{i_\mu}) = 0, \mu = 1, \dots, n$$

Putting $x = x_{i_\mu}$, where $\mu = 1, \dots, n$:

$$\Rightarrow \alpha_\mu = 0, \mu = 1, 2, \dots, n.$$

Thus $\{f_i; i \in I\}$ is a linearly independent subset

of L^* . So it can be extended to form a basis for L^* . Let B^* be a Hamel basis of L^* which contains the set $\{f_i : i \in I\}$.

Let $\{\beta_i : i \in I\}$ be a set of scalars such that $\beta_i \neq 0$ for infinitely many indices i . Define $f^* \in L^{**}$ by setting $f^*(f_i) = \beta_i$ and $f^*(f) = 0$ if $f \in B^*$ but f is not one of the elements f_i .

We shall show that f^* is not in the range of J i.e. there exists no element $\alpha \in L$ such that $J(\alpha) = f^*$.

Let $\alpha \in L$ be such that $J(\alpha) = f^*$.

Then by def. of J , we have $f^* = f_\alpha$. Therefore

$f^*(f_i) = f_\alpha(f_i) = f_i(\alpha) = \alpha_i$, where α_i is the coefficient of α_i in the representation of α in terms of the Hamel basis B . Now $\alpha_i = 0$ for all but a finite number of indices i . Therefore $f^*(f_i) = 0$ for all but a finite number of indices i . But according to our def. of f^* , we have $f^*(f_i) \neq 0$ for infinitely many indices i . Thus we get a contradiction.

Hence there exists no $\alpha \in L$ such that $J(\alpha) = f^*$.

Therefore the mapping J is not onto L^{**} . i.e. range of $J \neq L^{**}$.
Hence L is not algebraically reflexive.

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